

Spontaneous decay of periodic magnetostatic equilibria

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In order to understand the conditions which lead a highly magnetized, relativistic plasma to become unstable, and in such cases how the plasma evolves, we study a prototypical class of magnetostatic equilibria where the magnetic field satisfies $\nabla \times \mathbf{B} = \alpha \mathbf{B}$, where α is spatially uniform, on a periodic domain. Using numerical solutions we show that generic examples of such equilibria are unstable to ideal modes (including incompressible ones) which are marked by exponential growth in the linear phase. We characterize the unstable mode, showing how it can be understood in terms of merging magnetic and current structures, and explicitly demonstrate its instability using the energy principle. Following the nonlinear evolution of these solutions, we find that they rapidly develop regions with relativistic velocities and electric fields of comparable magnitude to the magnetic field, liberating magnetic energy on dynamical timescales and eventually settling into a configuration with the largest allowable wavelength. These properties make such solutions a promising setting for exploring the mechanisms behind extreme cosmic sources of gamma rays.

Introduction.—Magnetic stability is a fundamental question in a range of fields from laboratory plasma physics, where it influences the viability of fusion devices [1]; to space physics, where it controls the structure of magnetic fields within stars and planets [2]. In high-energy astrophysics, the spontaneous release of energy associated with transitions between magnetic equilibrium states is of particular importance to understanding the dramatic gamma-ray activities from pulsar wind nebulae [3, 4], magnetars [5–8], relativistic jets associated with active galactic nuclei [9–12], and gamma-ray bursts. These diverse sources exhibit powerful gamma-ray flares on timescales short compared with their light-crossing times [7, 11, 12], and seem to require that electrons and positrons be accelerated throughout extended regions, to energies as high as several PeV [13, 14]. The most dramatic variations are likely produced in the relativistic electromagnetic outflows away from the central engine (neutron stars or black holes), and a mechanism is pressing needed to explain the rapid, volumetric conversion of magnetic energy into high energy particles and radiation. Here, we consider whether such a process may be triggered by magnetic instability in the outflow. These outflows may initially accelerate, so that they cannot be crossed by hydromagnetic waves in an outflow timescale. However, they will eventually be decelerated when their momentum flux decreases to that of the external medium, bringing disconnected regions back into causal contact where they are likely to be unstable¹.

To understand the conditions under which a plasma becomes unstable, and to follow its subsequent nonlinear evolution in an idealized setting, we focus on a model class of force-free equilibria, which we find evolves in a

manner that is both surprising on formal grounds, and highly suggestive of the behavior of the most dramatic cosmic sources. Force-free solutions, where the Lorentz force vanishes, are an excellent approximation for highly conducting and strongly magnetized plasmas, where the plasma inertia and pressure is sub-dominant to the magnetic field, and have been used extensively across different fields. A particularly important class of force-free equilibria that are conjectured to arise naturally from magnetic relaxation are the so called Taylor states, which satisfy the Beltrami property: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ where α is a constant [15]. These solutions have played an important role not only in laboratory plasma physics [16], but also in solar physics [17–19], astrophysics [20], and beyond [21]. In this work we focus on space-periodic equilibria as a simple, computationally tractable setting free of the effect of confining boundaries (as in extended outflows). Though there is a rich literature studying such solutions [15, 22–25], important facts regarding their stability have not been appreciated. Focusing on a prototypical example, the “ABC” solutions [26] (defined below), in [24] it was claimed that such solutions are stable to incompressible perturbations (see also [25]). Here we show that, in fact, generic periodic Beltrami magnetic fields are linearly unstable, including to incompressible deformations. The only exceptions we find are special cases lacking magnetic curvature, and those in the fundamental mode or *ground state*, having the lowest magnetic energy compatible with conservation of magnetic helicity $H_M = \int \mathbf{A} \cdot \mathbf{B} dV$ (where \mathbf{A} is the magnetic vector potential). The instability we find is ideal, in contrast to previous studies of dissipative effects [27]. We find that in the nonlinear evolution, magnetic energy is indeed liberated rapidly, giving rise to relativistic velocities and electric fields of comparable magnitude to the magnetic fields on dynamical timescales, and eventually allowing the system to relax to its ground state. These solutions are therefore a simple, but promising setting

¹ A cosmological analogy would be when perturbations from the epoch of inflation are believed to have “re-entered” the horizon and exhibited gravitational instability.

to explore the mechanisms underlying extreme cosmic sources of gamma rays.

In what follows, we present simulation results showing the linear-regime instability of a range of magnetostatic equilibria, and then illustrate the properties of the dominant unstable mode in some example cases, independently confirming the growth rate using the energy principle. We then compare the results found using various degrees of magnetization, discuss the nonlinear evolution of the instability, and conclude. We use units with $c = 1$ throughout.

Methodology.—The equilibrium magnetic fields we study are exemplified by the three-parameter “ABC” field [26, 28] given by

$$\mathbf{B}^E = (B_3 \cos \alpha z - B_2 \sin \alpha y, \quad (1) \\ B_1 \cos \alpha x - B_3 \sin \alpha z, \quad B_2 \cos \alpha y - B_1 \sin \alpha x).$$

We use some particular examples of this equilibrium solution for illustrative purposes, but also consider the more general class of Beltrami fields [29] $\mathbf{B} = \alpha \mathbf{\Psi} + \nabla \times \mathbf{\Psi}$ where the potential field $\mathbf{\Psi}$ is any solenoidal vector field satisfying the vector Helmholtz equation $\nabla^2 \mathbf{\Psi} + \alpha^2 \mathbf{\Psi} = 0$, so that $\mathbf{\Psi}$ comprises only the Fourier harmonics whose wave-vector \mathbf{k} has magnitude α . These more general configurations are constructed by choosing random vector amplitudes for the admissible harmonics. Our computational domain is the periodic cube of length $L = 2\pi$ (though we restore L in some places for clarity).

We simulate a perfectly conducting, magnetized fluid and consider cases with different finite values of the volume-averaged magnetization parameter $\sigma := \langle B^2 / 4\pi \rho h \rangle$ where ρh is the fluid enthalpy (treated using the ideal relativistic magnetohydrodynamic equations and the code in [30]), as well as the limiting case of a completely magnetically dominated plasma, $\sigma = \infty$ (treated by force free electrodynamics [31–33]). See *supplemental material* below for details.

Instability in the Linear Regime.—For this class of magnetic equilibria, we find generic solutions with $\alpha^2 > 1$ to be unstable to linear ideal perturbations that are characterized by exponential growth of the electric field. Fig. 1 illustrates this for a case with $\alpha^2 = 11$. (Here we present results from $\sigma = \infty$ simulations, and in a later section we compare these to finite magnetization cases.) The magnitude of the growing solution is proportional to the initial perturbation², consistent with a linear instability. The growth rate of, e.g., the electric field energy is converging to $\gamma \approx 4.0\alpha/L$ with increased resolution, evidence that the instability is not due to numerical/non-ideal effects.

² Here the initial perturbation we use is an electric field with $E^x = E_0 \cos(2\pi y/L)$ and where the other components are given by cyclic permutations of $\{x, y, z\}$. From this we subtract out any component parallel to \mathbf{B} .

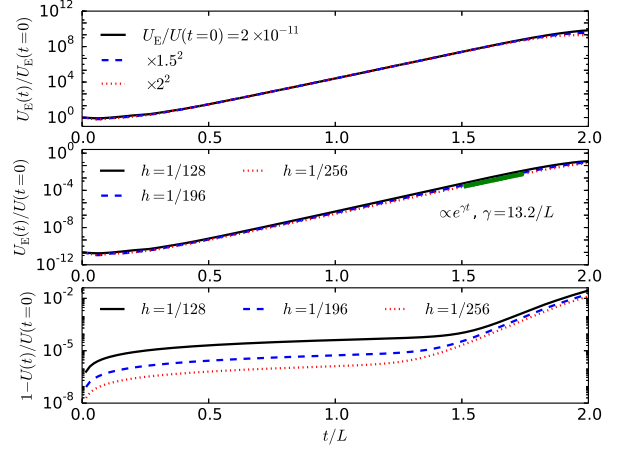


FIG. 1. Results from simulations with $\alpha^2 = 11$. Top: The growth of the electric field energy U_E , normalized by its initial value, for three different values of the initial perturbation. Middle: The growth in U_E for three different resolutions, along with an exponential fit. The difference between the best fit exponent for the high resolution, and the Richardson extrapolated value using all three resolutions, is $\approx 0.1\%$, the extrapolation being consistent with between first and second-order convergence. The bottom panel illustrates the conservation of total energy U for three different resolutions. Though initially higher-order when the equilibrium-solution truncation error dominates, the convergence eventually drops to first-order, presumably because (as discussed below) the unstable solution has non-smooth features. Conservation of magnetic helicity is similar.

Other equilibrium solutions exhibit similar exponentially growing solutions, as shown for some example cases in Fig. 2. This holds for wavelengths larger than the fundamental mode for the domain, $\alpha^2 = 1$, which is known to be stable [24, 25]. The growth rate of the instability is also roughly proportional to α , though there is dependence on the particular realization used.

The Dominant Unstable Mode.—In order to illustrate the nature of the instability, we focus on a simple type of $\alpha = 2$ equilibrium solution given by Eq. 1. We illustrate this solution for three different choices of coefficients in Fig. 3. As discussed in [28] for the mathematically equivalent Euler flow, these solutions have a rich structure. The $(B_1, B_2, B_3) = (1, 1, 0)$ case consists of “vortices:” regions of helical field (and current) lines circling a central axis. The $(B_1, B_2, B_3) = (1, 1/2, 0)$ case has vortices as well as “shear layers:” wavy field lines that begin and end on opposite sides of the domain. In addition to these two cases with z -translational symmetry, we also show a more generic case where all three coefficients are nonzero (and that like the second case, has no places where $\mathbf{B}^E = 0$).

In Fig. 3 we also show the corresponding velocity field $\mathbf{v} = \mathbf{E} \times \mathbf{B}^E / |\mathbf{B}^E|^2$ (which will be proportional to the dis-

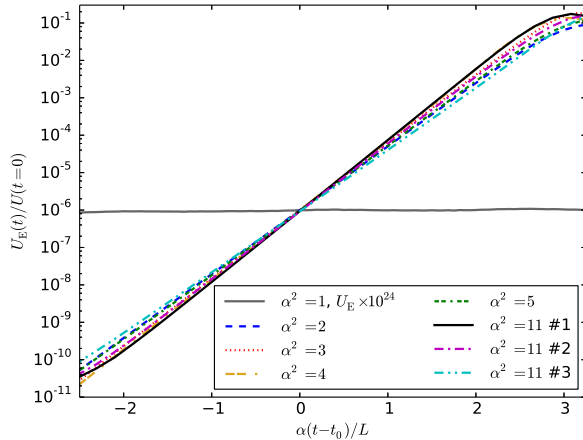


FIG. 2. The growth in electric field energy for various values of α^2 and three different realizations for $\alpha^2 = 11$. Time has been shifted so that all the curves have an abscissa of 0 at the ordinate value of 10^{-6} , and the time axis has been scaled by α which gives the different examples roughly the same slope. The $\alpha^2 = 1$ simulation does not exhibit exponential growth, and has been scaled up by an overall factor.

placement ξ for an eigenmode) characterizing the dominant instability arising in each case. This is calculated from a numerical snapshot after the instability has grown by roughly 10 orders of magnitude — seeded in this case just by truncation error — but is still in the linear regime ($|\mathbf{E}| \sim 10^{-4}|\mathbf{B}^E|$). The velocity field acts to bring together vortices, or current channels, circulating in the same direction in order to move towards a larger wavelength, lower magnetic energy configuration. The nonlinear evolution is characterized by the merging of magnetic vortices and can thus be related to the coalescence instability of magnetic islands [38]. We can also see that the velocity field appears to have non-smooth features, reminiscent of spontaneous current sheets [39], that occur at the separatrices dividing the vortices and shear layers. Though the generic case lacks z -translational symmetry, it appears qualitatively similar to the second case.

From the $(B_1, B_2, B_3) = (1, 1, 0)$ to the $(1, 1/2, 0)$ case, the growth rate of the instability decreases by a factor of ≈ 1.9 with the addition of the shear layers in the equilibrium solution. In fact, the growth rate decreases monotonically with B_2 , and as $B_2 \rightarrow 0$ and the vortices shrink to zero volume, the growth rate of the instability also goes to zero. In fact, it can be shown (see supplemental material below) that the single mode solutions are all stable, including those at short wavelengths.

For the generic case where the three coefficients are nonzero, we directly confirm that the instability is linear and ideal by using the energy principle [40]. This implies that an equilibrium solution \mathbf{B}^E satisfying the Beltrami

property is unstable to a displacement ξ if

$$\omega^2 := \frac{\int [|\delta\mathbf{B}|^2 - \alpha(\xi \times \mathbf{B}^E) \cdot \delta\mathbf{B}] dV}{\int |\mathbf{B}^E|^2 |\xi|^2 dV} < 0, \quad (2)$$

where $\delta\mathbf{B} := \nabla \times (\xi \times \mathbf{B}^E)$, and that the instability should grow at least as fast as $|\omega|$. Computing this quantity using finite differences, with the numerical velocity field, gives a value of $|\omega|L \approx 2.4$, near the growth rate of 2.6 measured for the electric field in the simulation.

Although the unstable displacement from the simulations has both non-smooth and compressible ($\int |\nabla \cdot \mathbf{v}| dV \approx 0.15 \int |\nabla \times \mathbf{v}| dV$) features, these are not necessary for instability. For example, by applying a low-pass filter to the Fourier harmonics of ξ one can construct an ideal perturbation that has no power at scales $|\mathbf{k}| > 29$, and yet gives a value of $|\omega|L \approx 2.3$ upon evaluating Eq. (2) algebraically in k -space. Furthermore, we have also explicitly constructed (see supplemental material below) analytical examples of smooth and incompressible displacements that destabilize particular Beltrami solutions, which are counterexamples to the claims in [24].

Magnetization comparison and nonlinear evolution.—

The same qualitative behavior is observed when the plasma is evolved with different magnetization parameters. Fig. 4 shows the evolution of the kinetic and magnetic energy U_B for different parameters σ values, including the limiting case of $\sigma = \infty$. The lower inset of Fig. 4 shows that the growth rate of the unstable mode increases monotonically with increasing σ and is roughly proportional to the Alfvén speed $v_A = \sqrt{\sigma/(1+\sigma)}$.

In both finite and infinite magnetization cases, exponential growth of the unstable displacement persists until its velocity $|\mathbf{v}|$ approaches the Alfvén speed v_A (near the speed of light for large σ) and higher-order evolutionary terms dominate. Following the turbulent state, the system settles into a lower energy equilibrium. Somewhat surprisingly, we *do not* observe direct evolution into the lowest energy ($\alpha = 1$) state for all cases. For example, the $\alpha^2 = 11$ state in Fig. 4 first transitions into a configuration with $\approx 97\%$ of its spectral energy in modes with $\mathbf{k}^2 = 3$, where it remains for about ten Alfvén crossing times, before making a second transition into the ground state, where 99% of its energy is in $\mathbf{k}^2 = 1$ modes. The lifetime of the intermediate state may be related to the fact that $\approx 88\%$ of the energy is in a single $\mathbf{k}^2 = 3$ mode.

During the nonlinear evolution, regions develop where $|\mathbf{E}|$ is comparable, and even exceeds $|\mathbf{B}|$. Since the hyperbolicity of the equations breaks down for $\sigma = \infty$ when this happens, to evolve further we handle such regions with an ad-hoc prescription where we simply reduce the electric-field magnitude to equal that of the magnetic field, leading to a reduction of energy. The finite σ cases do not suffer from this problem, and our scheme explicitly conserves total energy, but permits conversion of magnetic or kinetic energy into internal energy, especially at

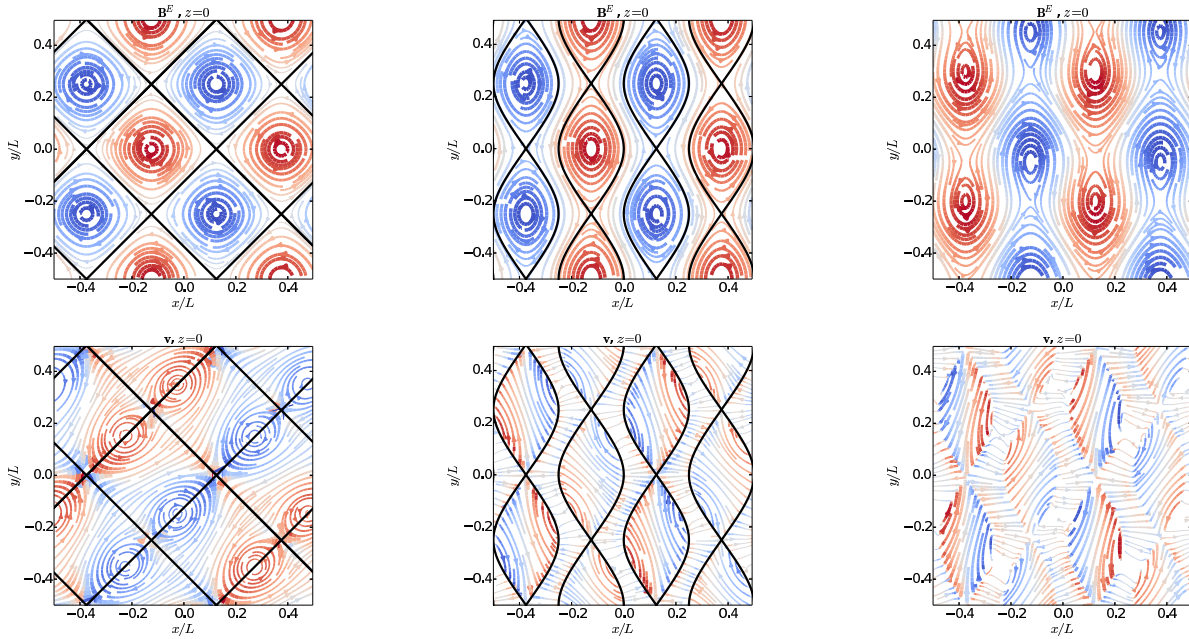


FIG. 3. Streamlines of a magnetic field equilibrium solution \mathbf{B}^E given by Eq. 1 with $\alpha = 2$ and various coefficients (top), and the corresponding velocity field $\mathbf{v} = \mathbf{E} \times \mathbf{B}^E / |\mathbf{B}^E|^2$ of the unstable mode arising from the simulations (bottom) in the $z = 0$ plane. The equilibrium solutions, from left to right, correspond to $(B_1, B_2, B_3) = (1, 1, 0)$, $(1, 1/2, 0)$, and $\approx (-0.814, 0.533, 0.232)$, respectively. The color indicates the perpendicular vector component with red and blue representing, respectively, out of the page and into the page. The thickness of the streamline is proportional to the vector magnitude. The black lines indicate the location of the separatrices in the equilibrium solutions.

shocks or places where the magnetic field is nearly discontinuous. Encouragingly, we still find consistency between these different (and somewhat arbitrary) types of energy dissipation in the non-linear regime. For example, as shown in Fig. 4, we find the same energy levels associated with the intermediate and final magnetic equilibria. This is consistent with conservation of magnetic helicity. Since the Beltrami fields have $\mathbf{B} = \alpha \mathbf{A}$, their helicity is $2U_B/\alpha$, and the ratio of magnetic energy in the α_i and α_f equilibria is simply α_f/α_i . Accordingly, we do not expect the dissipation mechanism to have much influence on the energy of the final state if helicity is preserved. (For the simulations shown in Fig. 4, H_M is constant to $\sim 0.1\%$.) But understanding the details of the energy dissipation will require better physical modeling.

Conclusions.—We studied periodic Beltrami magnetic fields and found that they were unstable to ideal modes. Though we focused on the relativistic case, since the instability is linear in velocity, it also applies to the nonrelativistic setting. This contrasts with [24] — where it was concluded that such solutions are linearly stable against incompressible perturbations — and suggests that the relaxation of a complex magnetic field will *not* terminate at a small wavelength equilibrium, but instead undergo a so-called inverse helicity cascade where magnetic energy goes to the largest available scale [41–44]. There

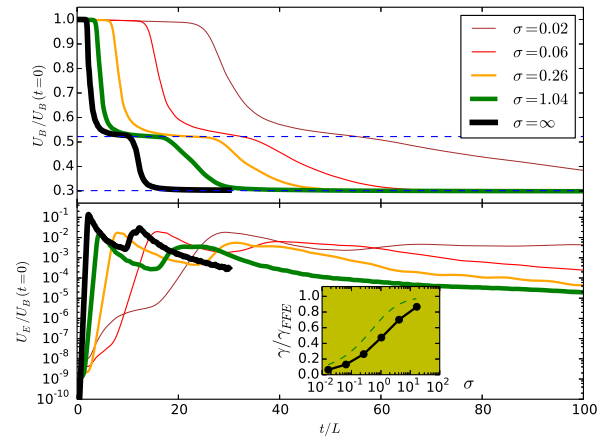


FIG. 4. A comparison of the decay of an $\alpha^2 = 11$ equilibrium in simulations with different values of magnetization parameter σ . Shown is the magnetic energy (top) and kinetic/electric field energy (bottom). The horizontal dashed lines in the top panel indicate the magnetic energy of $\alpha^2 = 3$ and $\alpha^2 = 1$ states with the same helicity. The bottom inset shows the linear growth rate γ measured for runs having different magnetization parameters, along with the Alfvén speed (dashed line) for comparison.

are known examples of unstable cylindrically-symmetric Beltrami solutions [45], and studying a broader class of geometries would be interesting follow-up work.

For highly magnetized, relativistic plasma, the instability gives rise to regions where the electric field magnitude is comparable to the magnetic field on dynamical timescales. In extreme cosmic sources of gamma rays, where such configurations may be relevant, these would be likely sites of particle acceleration and photon emission. Understanding the details of this, including the role of magnetic reconnection [46–48] and turbulence [49] in ultimately dissipating energy, and determining the nature of the acceleration mechanism [50–52], will require kinetic simulations incorporating radiative losses, something we plan for future work.

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Supplemental material: Details of numerical methods

In the main text we assume a perfectly conducting medium and perform simulations of cases having various degrees of magnetic dominance. The limiting case of a completely magnetically dominated plasma is treated by FFE [31–33]. The evolution equations of FFE are just the Maxwell equations with a prescription for the current derived from the assumption that the Lorentz force vanishes and can be written as [33, 34] (we use Heaviside-Lorentz units and set $c = 1$ throughout):

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (3a)$$

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mathbf{j}, \quad (3b)$$

$$\mathbf{j} = \frac{1}{B^2} [(\mathbf{B} \cdot (\nabla \times \mathbf{B}) - \mathbf{E} \cdot (\nabla \times \mathbf{E})) \mathbf{B} + (\nabla \cdot \mathbf{E}) \mathbf{E} \times \mathbf{B}]. \quad (3c)$$

We numerically solve these using fourth-order finite difference stencils and Runge-Kutta time stepping. We use hyperbolic divergence cleaning to exponentially damp violations of the $\nabla \cdot \mathbf{B} = 0$ constraint as in [35]. The $\mathbf{E} \cdot \mathbf{B} = 0$ constraint is explicitly enforced by redefining $\mathbf{E} \rightarrow \mathbf{E} - \mathbf{B}(\mathbf{E} \cdot \mathbf{B})/B^2$ at every coarse time step. We apply standard sixth-order Kreiss-Oliger [36] numerical dissipation to all the hyperbolic variables to suppress high frequency numerical error. The FFE code is parallelized using the PAMR/AMRD software library³.

We also solve the ideal relativistic magnetohydrodynamic (RMHD) equations (with a $\Gamma = 4/3$ equation of state) for different values of the volume-averaged magnetization parameter $\sigma := \langle B^2/4\pi\rho h \rangle$ (where ρh is the fluid enthalpy). In addition to the specified magnetic field, we use a restmass density and pressure that are initially equal and uniform. We use a second-order, constrained-transport, finite-volume scheme that explicitly conserves mass, energy, momentum, and magnetic flux. Full details of the code are described in [30].

Supplemental material: Unstable modes with analytical methods

In this sections, we verify the existence of unstable ideal modes for periodic Beltrami solutions using analytical methods. Introducing $\mathbf{v} = \mathbf{E} \times \mathbf{B}/B^2$ and $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, the evolution equations (3) can be rewritten in terms of the following equations [37]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (4a)$$

$$\frac{\partial}{\partial t} (B^2 \mathbf{v}) = (\nabla \times \mathbf{B}) \times \mathbf{B} + (\nabla \times \mathbf{E}) \times \mathbf{E} + (\nabla \cdot \mathbf{E}) \mathbf{E} \quad (4b)$$

³ <http://laplace.physics.ubc.ca/Group/Software.html>

Now consider small perturbation on a static equilibrium state which satisfies $\nabla \times \mathbf{B}_0 = \alpha \mathbf{B}_0$ with α being a constant. Let $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, and define the displacement field $\vec{\xi}$ such that to first order $\partial \vec{\xi} / \partial t = \mathbf{v}$. Suppose the perturbation can be decomposed into normal modes $\propto e^{i\omega t}$. The linearized version of Equation (4) now becomes

$$\mathbf{B}_1 = \nabla \times (\vec{\xi} \times \mathbf{B}_0), \quad (5a)$$

$$\begin{aligned} -\omega^2 B^2 \vec{\xi} &= (\nabla \times \mathbf{B}_1 - \alpha \mathbf{B}_1) \times \mathbf{B}_0 \\ &= (\nabla \times \mathbf{B}_0) \times (\nabla \times (\vec{\xi} \times \mathbf{B}_0)) + [\nabla \times (\nabla \times (\vec{\xi} \times \mathbf{B}_0))] \times \mathbf{B}_0 \\ &\equiv \hat{\mathbf{K}} \cdot \vec{\xi} \end{aligned} \quad (5b)$$

In the equation for $\vec{\xi}$ the differential operator $\hat{\mathbf{K}}$ is self-adjoint under periodic boundary conditions [40], so we can define a potential energy

$$V = -\frac{1}{2} \int d^3x \vec{\xi} \cdot \hat{\mathbf{K}} \cdot \vec{\xi} = \frac{1}{2} \int d^3x \left\{ [\nabla \times (\vec{\xi} \times \mathbf{B})]^2 - \alpha (\vec{\xi} \times \mathbf{B}) \cdot [\nabla \times (\vec{\xi} \times \mathbf{B})] \right\} \quad (6)$$

As a result

$$\omega^2 = \frac{-\int d^3x \vec{\xi} \cdot \hat{\mathbf{K}} \cdot \vec{\xi}}{\int d^3x B^2 \vec{\xi}_\perp \cdot \vec{\xi}_\perp} = \frac{V}{\frac{1}{2} \int d^3x B^2 (\vec{\xi}_\perp)^2}. \quad (7)$$

This allows a variational approach to the stability problem: we can use trial functions to get an upper limit on the lowest ω^2 . An equilibrium state is unstable as long as we can find one trial function that renders V negative. In the following we show a few examples.

1D equilibria

One-dimensional force-free equilibria in periodic box can be written as $\mathbf{B}_0 = (\cos \psi(z), \sin \psi(z), 0)$, where $\psi(z) = \alpha z$ for linear force-free fields. We can write the normal mode perturbation as $\vec{\xi} = \vec{\xi}(z) e^{i(k_x x + k_y y - \omega t)}$. Since the component of $\vec{\xi}$ parallel to the background \mathbf{B}_0 does not have physical significance, we can impose the requirement $\vec{\xi} \perp \mathbf{B}_0$. Then from Equation (5), we get

$$\xi_x(z) = -\frac{i \sin \psi (k_y \cos \psi - k_x \sin \psi)}{k_x^2 + k_y^2 - \omega^2} \xi'_z(z), \quad (8a)$$

$$\frac{1}{\omega^2 - (k_x \cos \psi + k_y \sin \psi)^2} \frac{d}{dz} \left((\omega^2 - (k_x \cos \psi + k_y \sin \psi)^2) \frac{d\xi_z}{dz} \right) + (\omega^2 - k_x^2 - k_y^2) \xi_z = 0 \quad (8b)$$

Eigensolutions with $\omega^2 > k_x^2 + k_y^2$ exist and they correspond to oscillating normal modes. However, we do not have normal modes with $\omega^2 < 0$: otherwise $\omega^2 - (k_x \cos \psi + k_y \sin \psi)^2 < 0$ and the solution would be essentially exponential so it cannot satisfy the boundary condition. As a result, the 1D equilibria with periodic boundary conditions are stable to ideal modes.

2D and 3D equilibria

In this case it's not realistic to solve the normal mode equation so we make use of the variational principle. In periodic box it's convenient to use Fourier basis to construct our parameterized trial functions: $\vec{\xi} = \sum_m \vec{\xi}_m e^{i\mathbf{k}_m \cdot \mathbf{x}}$, where $\mathbf{k}_{-m} = -\mathbf{k}_m$, $\vec{\xi}_{-m} = \vec{\xi}_m^*$ to ensure reality. \mathbf{B}_0 can also be decomposed into Fourier components: $\mathbf{B}_0 = \sum_n \mathbf{B}_n e^{i\mathbf{a}_n \cdot \mathbf{x}}$, where $\mathbf{a}_{-n} = -\mathbf{a}_n$, $\mathbf{B}_{-n} = \mathbf{B}_n^*$, $\mathbf{a}_n \cdot \mathbf{B}_n = 0$, $|\mathbf{a}_n| = |\alpha|$ and $i\mathbf{a}_n \times \mathbf{B}_n = \alpha \mathbf{B}_n$. Then the integrals for calculating the

potential energy in Equation (7) only involve algebraic manipulations (cf [24], with modifications):

$$\begin{aligned}
V &= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \mathbf{B}_1 \cdot [\mathbf{B}_1 - \vec{\xi} \times (\nabla \times \mathbf{B}_0)] dx dy dz \equiv \langle \mathbf{B}_1 \cdot [\mathbf{B}_1 - \vec{\xi} \times (\nabla \times \mathbf{B}_0)] \rangle \\
&= \sum_{\substack{\mathbf{K}=\mathbf{k}_m+\mathbf{a}_n \\ =\mathbf{k}'_m+\mathbf{a}'_n}} \sum_n \sum_{n'} [(\mathbf{k}_{m'} + \mathbf{a}_{n'}) \times (\vec{\xi}_{m'}^* \times \mathbf{B}_{n'}^*)] \cdot [(\mathbf{k}_m + \mathbf{a}_n) \times (\vec{\xi}_m \times \mathbf{B}_n) - \vec{\xi}_m \times (\mathbf{a}_n \times \mathbf{B}_n)] \\
&= \sum_{\mathbf{K}=\mathbf{k}_m+\mathbf{a}_n} \left\{ \frac{1}{2} \left| \sum_n [(\mathbf{k}_m \cdot \mathbf{B}_n) \vec{\xi}_m - (\mathbf{k}_m \cdot \vec{\xi}_m) \mathbf{B}_n] \right|^2 - \frac{1}{2} \left| \sum_n (\vec{\xi}_m \cdot \mathbf{B}_n) \mathbf{a}_n \right|^2 - \frac{1}{2} \left| \sum_n (\vec{\xi}_m \cdot \mathbf{a}_n) \mathbf{B}_n \right|^2 \right. \\
&\quad \left. + \frac{1}{2} \left| \sum_n [(\mathbf{k}_m \cdot \mathbf{B}_n) \vec{\xi}_m - (\mathbf{k}_m \cdot \vec{\xi}_m) \mathbf{B}_n - (\vec{\xi}_m \cdot \mathbf{B}_n) \mathbf{a}_n - (\vec{\xi}_m \cdot \mathbf{a}_n) \mathbf{B}_n] \right|^2 \right\} \quad (9)
\end{aligned}$$

For certain equilibria this expression is not positive definite and can turn out to be negative using appropriate trial functions.

In the following we focus on a particular case, the ABC field

$$\mathbf{B} = B_1(0, \cos \alpha x, -\sin \alpha x) + B_2(-\sin \alpha y, 0, \cos \alpha y) + B_3(\cos \alpha z, -\sin \alpha z, 0), \quad (10)$$

which has been studied in the main text using numerical simulations.

The first case is 2D with $B_1 = B_2 = 1$, $B_3 = 0$, $\alpha = 2 > 1$. We include several of the longest wavelength perturbations in the trial function: $\mathbf{k} = (1, 1, 0)$, $(1, -1, 0)$, $(1, 3, 0)$, $(1, -3, 0)$, $(3, 1, 0)$, $(3, -1, 0)$, $(3, 3, 0)$, $(3, -3, 0)$ plus their negative companions, then decide the coefficients $\vec{\xi}_{\mathbf{k}}$ by minimizing the right hand side of Equation (7). The \mathbf{k} 's are chosen based on the observation that in the numerical simulation, the dominant unstable mode is 2D and has an electric field $\mathbf{E} \propto \vec{\xi} \times \mathbf{B}$ mainly comprised of $\mathbf{k} = (\pm 1, \pm 1, 0)$ components. We find that the minimization does give negative ω^2 : $\omega^2 < -0.04$, meaning a growth rate of $\gamma > 0.2$. As a comparison, the light crossing time scale $\tau = 2\pi$ so $\gamma\tau > 1.26$. This growth rate is less than what has been found from numerical simulations ($\gamma\tau \approx 5.5$ for this particular case) which is to be expected since we can only get a lower limit on the growth rate from a variational approach. The Fourier components of this trial function are listed in Table I, and Figure 5 shows the stream plot of the perturbation. We find that including higher \mathbf{k} 's (but only $\mathbf{k} = (\pm(2s+1), \pm(2t+1), 0)$, $s, t \in \mathbb{Z}$ are relevant) allows us to get lower minimum ω^2 (i.e. larger growth rate). For example, when the Fourier modes for the perturbation $\vec{\xi}$ include the following: $\mathbf{k} = (1, 1, 0)$, $(1, -1, 0)$, $(1, 3, 0)$, $(1, -3, 0)$, $(3, 1, 0)$, $(3, -1, 0)$, $(3, 3, 0)$, $(3, -3, 0)$, $(1, 5, 0)$, $(1, -5, 0)$, $(5, 1, 0)$, $(5, -1, 0)$, $(3, 5, 0)$, $(3, -5, 0)$, $(5, 3, 0)$, $(5, -3, 0)$, $(5, 5, 0)$, $(5, -5, 0)$, we get the growth rate $\gamma > 0.38$ and $\gamma\tau > 2.38$. The corresponding trial function is plotted in Figure 6. Thus, the short wavelength perturbations are essential for the instability. Comparing these analytical trial functions with the dominant unstable mode discovered in numerical simulations, we find that the former is consistent with being truncated version of the latter.

TABLE I. Fourier components of the unstable trial function (compressible) for the equilibrium $B_1 = B_2 = 1$, $B_3 = 0$, $\alpha = 2$

\mathbf{k}_m^\dagger	$\Re(\vec{\xi}_m)$	$\Im(\vec{\xi}_m)$
$\{1, 1, 0\}$	$\{0.0795, 0.0688, 0.0641\}$	$\{0.1367, 0.0116, 0.0502\}$
$\{1, -1, 0\}$	$\{-0.0623, -0.0569, 0.1211\}$	$\{-0.1611, -0.1557, 0.0766\}$
$\{1, 3, 0\}$	$\{-0.0646, 0.0145, 0.0474\}$	$\{0.0342, 0.0145, 0.0515\}$
$\{1, -3, 0\}$	$\{-0.0400, 0.0045, 0.1013\}$	$\{-0.0972, 0.0045, -0.0441\}$
$\{3, 1, 0\}$	$\{-0.0145, 0.0700, -0.0420\}$	$\{0.0145, 0.0289, 0.0568\}$
$\{3, -1, 0\}$	$\{0.0045, 0.1083, 0.0470\}$	$\{-0.0045, -0.0511, 0.1042\}$
$\{3, 3, 0\}$	$\{0.0048, -0.0048, 0.0008\}$	$\{-0.0048, 0.0048, 0.0008\}$
$\{3, -3, 0\}$	$\{-0.0154, -0.0154, 0.0027\}$	$\{0.0154, 0.0154, 0.0027\}$
$\omega^2 \leq -0.04$, $\tau = 2\pi$, $\omega\tau \geq 1.26$		

[†] The coefficients for the $-\mathbf{k}_m$ terms are just the complex conjugate of the listed \mathbf{k}_m coefficients.

We also find that the instability still exist when incompressibility is imposed, i.e. $\nabla \cdot \vec{\xi} = 0$. Table II lists the Fourier components for the incompressible trial function that gives negative potential energy and Figure 7 shows the corresponding stream plot.

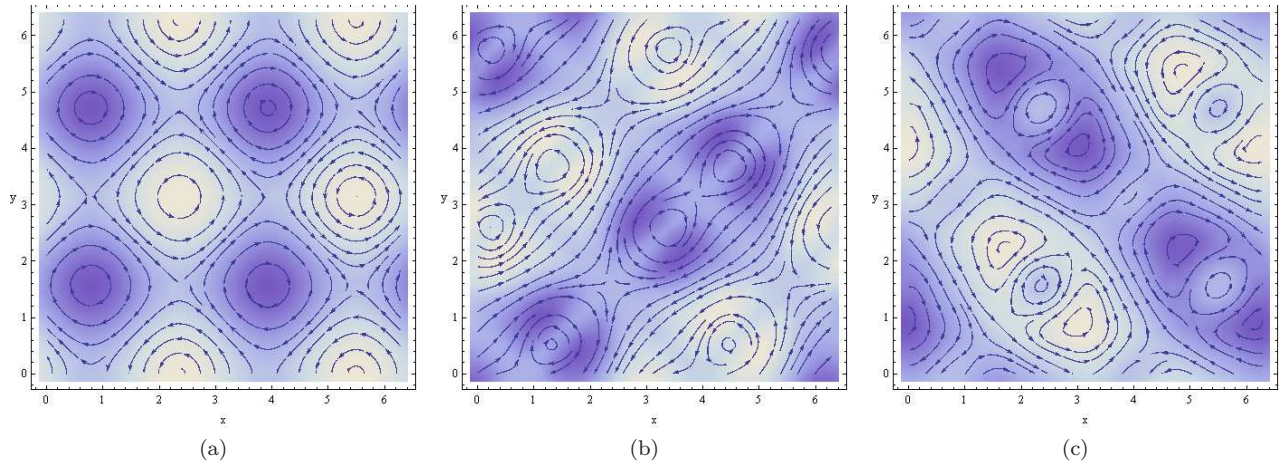


FIG. 5. (a) The equilibrium ABC field (10) with $B_1 = B_2 = 1$, $B_3 = 0$, $\alpha = 2 > 1$. The stream lines indicate the field components in the $x - y$ plane and color indicates the component perpendicular to the plane (same below for other vector fields). (b) Trial perturbation $\vec{\xi}$ that renders negative potential energy (instability) for the equilibrium. We only plot $\vec{\xi}_\perp$, the components perpendicular to the equilibrium magnetic field. The perturbation is compressible in this case. (c) Perturbation magnetic field \mathbf{B}_1 resulted from the perturbation in (b).

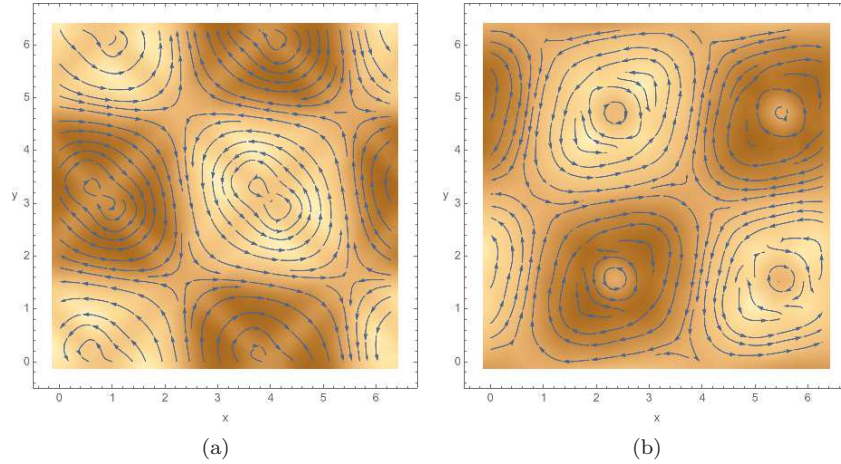


FIG. 6. (a) Similar to Figure 5 (b), this is the displacement field $\vec{\xi}_\perp$ derived from a trial function involving more Fourier modes. (b) Perturbation magnetic field \mathbf{B}_1 resulted from the perturbation in (a).

Other more general 2D cases with $B_1 \neq B_2$ and $B_1 B_2 \neq 0$ are found to be unstable as well in numerical simulations, with growth rate decreasing from maximum to 0 as, say, B_2/B_1 goes from 1 to 0. As an illustration we apply our variational principle to the case $B_1 = 1$, $B_2 = 1/2$, $B_3 = 0$, using trial functions comprised of Fourier modes $\mathbf{k} = (1, 1, 0)$, $(1, -1, 0)$, $(1, 3, 0)$, $(1, -3, 0)$, $(3, 1, 0)$, $(3, -1, 0)$, $(3, 3, 0)$, $(3, -3, 0)$ and their negative companions. This is shown in Table III Figure 8, and the displacement field can be readily compared with Figure 3 in the main text. The growth rate we get from this trial function is $\gamma\tau > 0.8$ — the lower limit has been reduced from corresponding $B_1 = B_2 = 1$ case.

It is also instructive to consider a 3D example. Here we take $B_1 = B_2 = 1$, $B_3 = 1/5$ and $\alpha = 2$. Still use Fourier components $\mathbf{k} = (1, 1, 0)$, $(1, -1, 0)$, $(1, 3, 0)$, $(1, -3, 0)$, $(3, 1, 0)$, $(3, -1, 0)$, $(3, 3, 0)$, $(3, -3, 0)$ in the trial function, we found the unstable perturbation as shown in Table IV and Figure 9.

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TABLE II. Fourier components of the unstable trial function (incompressible) for the equilibrium $B_1 = B_2 = 1$, $B_3 = 0$, $\alpha = 2$

\mathbf{k}_m^\dagger	$\Re(\vec{\xi}_m)$	$\Im(\vec{\xi}_m)$
$\{1, 1, 0\}$	$\{-0.1007, 0.1007, 0.0005\}$	$\{-0.1007, 0.1007, -0.0006\}$
$\{1, -1, 0\}$	$\{-0.0600, -0.0600, -0.0003\}$	$\{-0.0600, -0.0600, 0.0004\}$
$\{1, 3, 0\}$	$\{-0.0108, 0.0036, 0.0001\}$	$\{-0.0108, 0.0036, 0\}$
$\{1, -3, 0\}$	$\{-0.0181, -0.0060, -0.0001\}$	$\{-0.0181, -0.0060, 0.0001\}$
$\{3, 1, 0\}$	$\{-0.0036, 0.0108, -0.0001\}$	$\{0.0036, -0.0108, 0\}$
$\{3, -1, 0\}$	$\{-0.0060, -0.0181, 0.0001\}$	$\{0.0060, 0.0181, 0.0001\}$
$\{3, 3, 0\}$	$\{-0.0139, 0.0139, 0\}$	$\{0.0139, -0.0139, 0\}$
$\{3, -3, 0\}$	$\{-0.0083, -0.0083, 0\}$	$\{0.0083, 0.0083, 0\}$
$\omega^2 \leq -0.016$, $\tau = 2\pi$, $\omega\tau \geq 0.795$		

[†] The coefficients for the $-\mathbf{k}_m$ terms are just the complex conjugate of the listed \mathbf{k}_m coefficients.

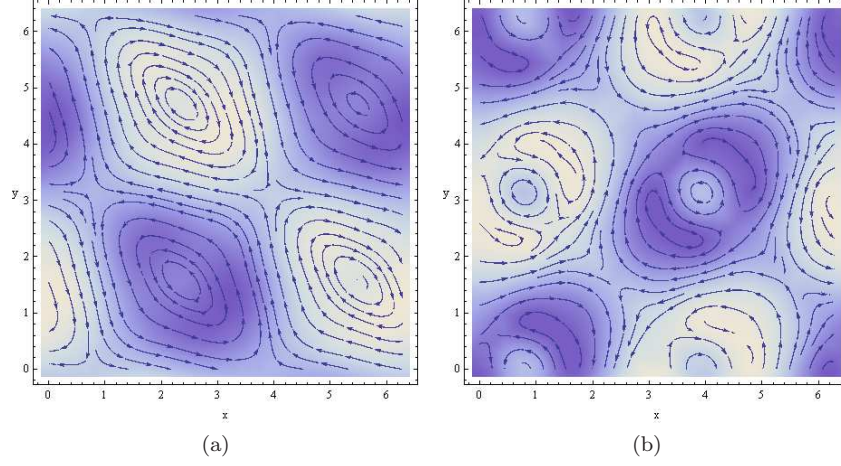


FIG. 7. (a) Incompressible trial perturbation $\vec{\xi}$ that renders negative potential energy (instability) for the equilibrium Figure 5(a). (b) Perturbation magnetic field \mathbf{B}_1 resulted from the perturbation in (a).

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TABLE III. Fourier components of the unstable trial function (compressible) for the equilibrium $B_1 = 1, B_2 = 1/2, B_3 = 0, \alpha = 2$

\mathbf{k}_m^\dagger	$\Re(\vec{\xi}_m)$	$\Im(\vec{\xi}_m)$
$\{1, 1, 0\}$	$\{0.0914, -0.1849, 0.0545\}$	$\{0.1094, -0.2210, -0.0004\}$
$\{1, -1, 0\}$	$\{-0.0117, -0.0513, -0.1015\}$	$\{0.0577, 0.0876, -0.1069\}$
$\{1, 3, 0\}$	$\{0.0306, -0.0022, -0.0378\}$	$\{-0.0389, -0.0022, -0.0317\}$
$\{1, -3, 0\}$	$\{0.0274, 0.0140, -0.0163\}$	$\{0.0093, 0.0140, 0.0343\}$
$\{3, 1, 0\}$	$\{0.0033, -0.0879, 0.0619\}$	$\{-0.0033, -0.0510, -0.0770\}$
$\{3, -1, 0\}$	$\{0.0209, 0.0455, -0.0593\}$	$\{-0.0209, -0.0094, -0.0233\}$
$\{3, 3, 0\}$	$\{0.0166, -0.0184, 0.0066\}$	$\{-0.0166, 0.0184, 0.0066\}$
$\{3, -3, 0\}$	$\{0.0026, 0.0029, -0.0010\}$	$\{-0.0026, -0.0029, -0.0010\}$
$\omega^2 \leq -0.017, \tau = 2\pi, \omega\tau \geq 0.809$		

[†] The coefficients for the $-\mathbf{k}_m$ terms are just the complex conjugate of the listed \mathbf{k}_m coefficients.

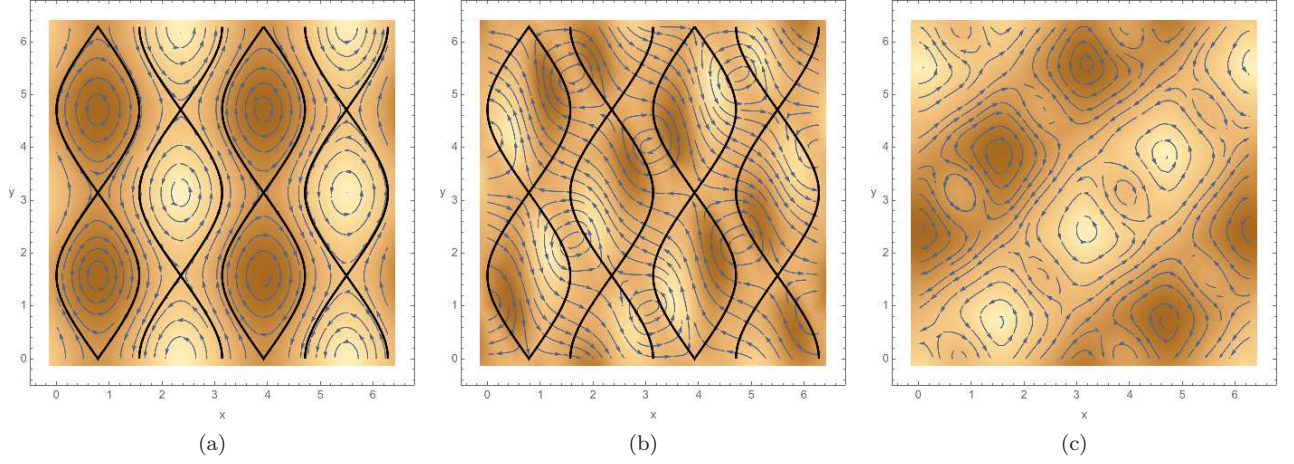


FIG. 8. (a) The equilibrium ABC field (10) with $B_1 = 1, B_2 = 1/2, B_3 = 0, \alpha = 2 > 1$. The stream lines indicate the field components in the $x-y$ plane and color indicates the component perpendicular to the plane (same below for other vector fields). The thick black lines show the separatrices — surfaces separating different topological domains — in the equilibrium magnetic field. (b) Trial perturbation ξ that renders negative potential energy (instability) for the equilibrium. We only plot ξ_\perp , the components perpendicular to the equilibrium magnetic field. The perturbation is compressible in this case. (c) Perturbation magnetic field \mathbf{B}_1 resulted from the perturbation in (b).

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TABLE IV. Fourier components of the unstable trial function (compressible) for the equilibrium $B_1 = B_2 = 1$, $B_3 = 1/5$, $\alpha = 2$

\mathbf{k}_m^\dagger	$\Re(\vec{\xi}_m)$	$\Im(\vec{\xi}_m)$
$\{1, 1, 0\}$	$\{-0.1122, 0.1121, -0.0231\}$	$\{-0.1121, 0.1120, 0.0233\}$
$\{1, -1, 0\}$	$\{-0.0180, -0.0180, 0.0039\}$	$\{-0.0180, -0.0180, -0.0036\}$
$\{1, 3, 0\}$	$\{-0.0029, 0.0024, -0.0007\}$	$\{-0.0029, 0.0024, 0.0008\}$
$\{1, -3, 0\}$	$\{-0.0182, -0.0147, 0.0048\}$	$\{-0.0182, -0.0147, -0.0047\}$
$\{3, 1, 0\}$	$\{-0.0024, 0.0029, 0.0007\}$	$\{0.0024, -0.0029, 0.0008\}$
$\{3, -1, 0\}$	$\{-0.0147, -0.0182, -0.0048\}$	$\{0.0147, 0.0183, -0.0047\}$
$\{3, 3, 0\}$	$\{-0.0151, 0.0151, -0.0026\}$	$\{0.0151, -0.0151, -0.0026\}$
$\{3, -3, 0\}$	$\{-0.0024, -0.0024, 0.0004\}$	$\{0.0024, 0.0024, 0.0004\}$
$\omega^2 \leq -0.016$, $\tau = 2\pi$, $\omega\tau \geq 0.801$		

[†] The coefficients for the $-\mathbf{k}_m$ terms are just the complex conjugate of the listed \mathbf{k}_m coefficients.

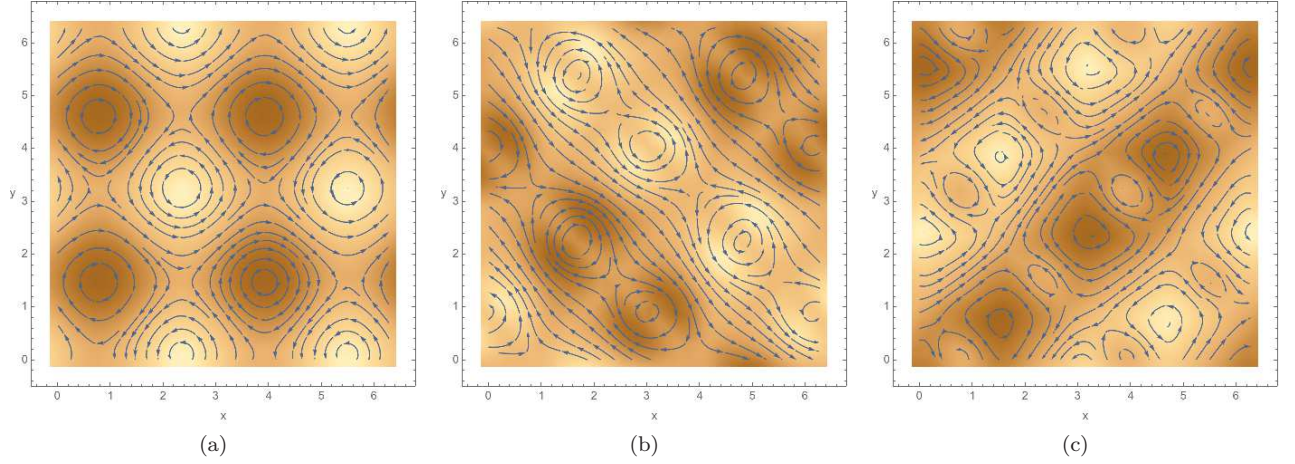


FIG. 9. (a) The equilibrium ABC field (10) with $B_1 = B_2 = 1$, $B_3 = 1/5$, $\alpha = 2 > 1$. The stream lines indicate the field components in the $z = 0$ plane and color indicates the component perpendicular to the plane (same below for other vector fields). (b) Trial perturbation $\vec{\xi}$ that renders negative potential energy (instability) for the equilibrium. We only plot $\vec{\xi}_\perp$, the components perpendicular to the equilibrium magnetic field. The perturbation is compressible in this case. (c) Perturbation magnetic field \mathbf{B}_1 resulted from the perturbation in (b). Both (b) and (c) are stream plots on the $z = 0$ plane.

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